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A NONSINGULAR UNIVERSE<sup>★</sup>

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## ABSTRACT

We construct an effective action for gravity in which all homogeneous solutions are nonsingular<sup>1)</sup>. In particular, there is neither a big bang nor a big crunch. The action is a higher derivative modification of Einstein's theory constructed in analogy to how the action for point particle motion of particles in special relativity is obtained from Newtonian mechanics.

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## 1. Introduction

The singularity theorems of general relativity<sup>2)</sup> prove that space–time manifolds in general relativity are geodesically incomplete. The key assumptions which are used in the proofs of these theorems are that the action of gravity is an unmodified Einstein action, and that matter satisfies the energy dominance condition  $\epsilon > 0$  and  $\epsilon + 3p \geq 0$ , where  $\epsilon$  and  $p$  are energy density and pressure respectively.

Whereas the singularity theorems do not provide any general information about the nature of the singularity, well known examples of space–time show that at a singularity typically some of the curvature invariants  $R$ ,  $R_{\mu\nu} R^{\mu\nu}$ ,  $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ ,  $\dots$  blow up. These singularities should be viewed as a breakdown of general relativity at high curvatures.

Quantum gravity and alternative fundamental theories of all four forces, such as string theory, have been invoked as ways towards a solution of the singularity problems. As yet these avenues have not been developed far enough to provide a solution. We believe, however, that a new fundamental theory which includes gravity will solve the singularity problem. One way towards realizing a singularity free theory is to guess an effective action for gravity which contains no singularities. Constructing such a theory is what we attempt in this work.

As a preliminary, recall that the well known and successful theories of special relativity (SR) and quantum mechanics are based on inequalities  $v < c$  and  $\Delta x \Delta p \geq \hbar$  respectively. Hence, the hope arises that it might be possible to construct a new theory of gravity based on an inequality involving Newton’s constant  $G$ . For example, in a theory with a fundamental length  $\ell_f$  (i.e.  $\ell \geq \ell_f$  for all lengths  $\ell$ ), all curvature invariants are automatically bounded ( $R \leq \ell_f^{-2}$ ,  $R_{\mu\nu} R^{\mu\nu} \leq \ell_f^{-4}$ , etc.). However, a fundamental length is incompatible with a continuum theory of space and time, and thus we will attempt to realize the constraints on the curvature invariants directly..

Our goal is to construct a theory in which all curvature invariants are bounded and in which space–time is geodesically complete. This formidable problem can be reduced substantially by invoking the “Limiting Curvature Hypothesis”<sup>3)</sup>, according to which one

- i) finds a theory in which a small number of invariants is explicitly bounded, and
- ii) when these invariants take on their limiting values, a definite nonsingular solution (namely de Sitter) is taken on.

As a consequence of the limiting curvature hypothesis, automatically all invariants are bounded, and space–time is geodesically complete in its asymptotic regions.

The limiting curvature hypothesis has interesting consequences for Friedmann models and for spherically symmetric vacuum space–times<sup>4)</sup>. A collapsing Universe will not reach a big crunch, but will end up as a contracting de Sitter Universe ( $k = 0$ ) or a de Sitter bounce ( $k = 1$ ) followed by re-expansion (see Fig. 1). For a spherically symmetric vacuum solution, there would be no singularity inside the Schwarzschild horizon; instead, a de Sitter Universe will be reached when falling through the horizon towards large curvature (see Fig. 1.).

**Fig. 1:** Penrose diagrams for a collapsing Universe (left) and for a black hole (right) in Einstein’s theory and after implementing the limiting curvature hypothesis (bottom). Wavy lines denote a singularity (in the case of the collapsing Universe the big crunch), the symbols C, DS and E stand for collapsing phase, de Sitter phase and expanding phase respectively, and H denotes the Schwarzschild horizon.

## 2. Construction

In order to realize the limiting curvature hypothesis, we must abandon at least one of the assumptions of the Penrose–Hawking theorems. Unlike in inflationary cosmology<sup>5)</sup> we do not invoke “strange” matter which violates the energy dominance condition. Instead, we drop the assumption that gravity is described by a pure Einstein action.

The theory discussed here is a higher derivative modification of Einstein gravity. It is reasonable to consider such modifications since the Einstein theory is known to break down at high curvatures – based on perturbative quantum gravity calculations, quantum field theory effects in curved space–time, and on taking low energy limits of fundamental theories of all forces such as string theory.

Most higher derivative gravity models have much worse singularity properties than the Einstein theory. Hence, it is a nontrivial task to construct a model which has better properties. As an added bonus, the construction which leads to our nonsingular Universe is well motivated in analogy to how the action for particle motion in special relativity emerges from the point particle action in Newtonian mechanics.

Special relativity is a theory in which point particle velocities  $v$  are bounded. Starting from Newtonian mechanics (in which  $v$  is unbounded) for which the point particle action is

$$S = m \int dt \frac{1}{2} \dot{x}^2, \quad (2.1)$$

$m$  being the particle mass, the action for special relativity can be obtained<sup>6)</sup> using a Lagrange multiplier construction

$$S = m \int dt \left[ \frac{1}{2} \dot{x}^2 + \varphi \dot{x}^2 - V(\varphi) \right]. \quad (2.2)$$

Provided that  $V(\varphi)$  increases no faster than  $\varphi$  at large  $\varphi$ , the quantity which couples to  $\varphi$ , namely  $\dot{x}^2$ , is automatically bounded, as follows from the variational equation with respect to  $\varphi$

$$\dot{x}^2 = \frac{\partial V}{\partial \varphi}. \quad (2.3)$$

In order to recover the correct Newtonian limit at low velocities,  $V(\varphi)$  must be proportional

to  $\varphi^2$  as  $\varphi \rightarrow 0$ . Thus, the conditions on  $V(\varphi)$  are

$$V(\varphi) \sim \begin{cases} \varphi & |\varphi| \rightarrow \infty \\ \varphi^2 & \varphi \rightarrow 0. \end{cases} \quad (2.4)$$

Up to factors of 2, the simplest potential which satisfies (2.4) is

$$V(\varphi) = \frac{2\varphi^2}{1 + 2\varphi} \quad (2.5)$$

Eliminating the Lagrange multiplier  $\varphi$  via (2.3) and substituting into (2.2), the action of a point particle in special relativity

$$S = \int dt \sqrt{1 - \dot{x}^2} \quad (2.6)$$

results.

Our idea<sup>1)</sup> is to imitate the above construction in gravity. Starting with Einstein's theory of general relativity with action

$$S = \int d^4x \sqrt{-g} R \quad (2.7)$$

and unbounded Ricci scalar curvature, we construct a new gravity theory by introducing a Lagrange multiplier  $\varphi_1$ , with potential  $V_1(\varphi_1)$  which couples to  $R$ , the quantity we wish to bound:

$$S = \int d^4x \sqrt{-g} [R + \varphi_1 R + V_1(\varphi_1)] . \quad (2.8)$$

The potential  $V_1(\varphi_1)$  must satisfy the same asymptotic properties as given in (2.4).

However, the action (2.8) is not sufficient. In order to obtain a nonsingular Universe, we must implement the limiting curvature hypothesis. This is achieved once again by using the Lagrange multiplier technique. At this point we restrict our attention for the moment to homogeneous and isotropic space-times.

Consider the invariant

$$I_2 = 4R_{\mu\nu} R^{\mu\nu} - R^2 . \quad (2.9)$$

This invariant is positive semidefinite, and vanishes only if space-time is de Sitter. Hence, we will implement the limiting curvature hypothesis by forcing  $I_2$  to zero at high curvatures.

We chose the action

$$S = \int d^4x \sqrt{-g} \left[ R + \varphi_1 R + \varphi_2 \sqrt{I_2} + V_1(\varphi_1) + V_2(\varphi_2) \right]. \quad (2.10)$$

Provided that

$$V_2(\varphi_2) \sim \begin{cases} \text{const} & |\varphi_2| \rightarrow \infty \\ \varphi_2^2 & \varphi_2 \rightarrow 0 \end{cases} \quad (2.11)$$

then for  $|\varphi_2| \rightarrow \infty$  space-time becomes de Sitter, and the low curvature limit of the theory agrees with general relativity.

By construction, a theory with action (2.10) becomes de Sitter at large  $\varphi_2$ . It remains to be shown that there are no singularities for finite values in the  $\varphi_1/\varphi_2$  phase space. To show this, we need a specific model.

### 3. Specific Model

As the most simple realization of a nonsingular Universe we consider the action<sup>1)</sup>

$$S = \int d^4x \sqrt{-g} \left[ (1 + \varphi_1)R - \left( \varphi_2 + \frac{6}{\sqrt{12}} \varphi_1 \right) I_2^{1/2} + V_1(\varphi_1) + V_2(\varphi_2) \right] \quad (3.1)$$

with

$$\begin{aligned} V_1(\varphi_1) &= 12H_0^2 \frac{\varphi_1^2}{1 + \varphi_1} \left( 1 - \frac{\ln(1 + \varphi_1)}{1 + \varphi_1} \right) \\ V_2(\varphi_2) &= -\sqrt{12} H_0^2 \frac{\varphi_2^2}{1 + \varphi_2^2} \end{aligned} \quad (3.2)$$

Apart from the logarithmic term in  $V_1$ , the above potentials are the most simple ones which satisfy the asymptotic conditions (2.4) and (2.11). It was necessary<sup>1)</sup> to add the next leading (logarithmic) term in  $V_1$  in order to prevent trajectories from reaching  $\varphi_1 \rightarrow \infty$  for  $|\varphi_2| < 1$ .

The general variational equations which follow from (3.1) are rather complicated (see Ref. 7). However, when applied to a collapsing Universe with metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \quad (3.3)$$

and Hubble parameter

$$H = \frac{\dot{a}}{a} < 0, \quad (3.4)$$

the variational equations become simple<sup>1)</sup>:

$$H^2 = \frac{1}{12} V_1', \quad (3.5)$$

$$\dot{H} = -\frac{1}{\sqrt{12}} V_2' \quad (3.6)$$

and

$$3(1 - 2\varphi_1) H^2 + \frac{1}{2} (V_1 + V_2) = \frac{6}{\sqrt{12}} H(\dot{\varphi}_2 + 3H\varphi_2). \quad (3.7)$$

From (3.5) it follows that  $\varphi_1 > 0$ , from (3.6) that  $|\varphi_2| \rightarrow \infty$  is equivalent to de Sitter space, and (3.7) can be combined with the time derivative of (3.5) and with (3.6) to yield

$$\frac{d\varphi_2}{d\varphi_1} = \frac{V_1''}{V_1' V_2'} \left[ -\frac{1}{4}(1 - 2\varphi_1) V_1' + \frac{1}{2} (V_1 + V_2) + \frac{3}{2\sqrt{12}} V_1' \varphi_2 \right], \quad (3.8)$$

an equation from which the trajectories of this dynamical system in  $\varphi_1/\varphi_2$  phase space can be read off.

The system of equations (3.5, 3.6 & 3.8) must be analyzed to show that there are no singular solutions. The asymptotic regions  $|\varphi_1|, |\varphi_2| \ll 1$  and  $|\varphi_1|, |\varphi_2| \gg 1$  can be analyzed analytically<sup>1)</sup>. It can be seen that there are two types of solutions: periodic solutions about Minkowski space ( $\varphi_1 = \varphi_2 = 0$ ) and solutions which start and end at  $|\varphi_2| = \infty$ , i.e. in de Sitter space (see Fig. 2). It can be shown numerically<sup>7)</sup> that there are indeed no singular points for finite values of phase space and that the trajectories connect in a way which can be guessed from the analytical analysis of the asymptotic regions. Thus, we have demonstrated that all solutions are nonsingular.

So far, only vacuum solutions of our new gravitational theory have been discussed. It is easy to include matter in the analysis by considering the action

$$S_{\text{full}} = S + S_m, \quad (3.9)$$

where  $S$  is the gravitational action of (3.1), and  $S_m$  is the action for matter in the presence of the metric  $g_{\mu\nu}$ . We have investigated<sup>7)</sup> the model obtained by adding hydrodynamical matter with an equation of state  $p = w\rho$  and  $w = 0$  (cold matter) or  $w = 1/3$  (radiation).

The interesting result of our analysis<sup>7)</sup> is that for  $|\varphi_2| \rightarrow \infty$  the trajectories are unchanged when adding matter, even though for a contracting spatially flat Universe the energy density is increasing exponentially.

The only change for a spatially closed Universe is that the contracting de Sitter phase is replaced by a de Sitter bounce.

In conclusion, we have presented a model in which all homogeneous and isotropic solutions are nonsingular.



#### 4. Wild Speculations

Since matter does not change the evolution of space–time at large curvatures, the gravitational interactions are asymptotically free, i.e. the effective coupling  $G_{\text{eff}}$  of matter to gravity tends to zero. This is a first very nice property of our model.

Secondly, when applied to an expanding Universe, our theory implies that it has emerged from an initial de Sitter phase. Thus, an inflationary period is obtained without assuming the presence of matter violating the energy dominance condition. This result, however, is no surprise, since it is well known<sup>8)</sup> that higher derivative gravity models lead to inflation.

Let us now combine the first two results and consider the quantum generation of density perturbations in the initial de Sitter phase. These perturbations are stretched by inflation and may become the seeds for structures in the Universe. In scalar field driven inflationary models, the magnitude of the scalar metric fluctuations is too large without requiring that a particle physics parameter (coupling constant of a  $\lambda\varphi^4$  interaction term or a mass scale  $m$  in a theory of chaotic inflation with potential  $\frac{1}{2}m^2\varphi^2$ ) be artificially small. However, since the magnitude of these perturbations is proportional to  $G_{\text{eff}}$ , it is conceivable that in our model there will be no fine tuning for inflation.

Next, let us consider an application to black holes. For black holes in Einstein’s theory of general relativity, Hawking radiation leads to its evaporation with ever increasing speed. However, the strength of Hawking radiation is proportional to  $G_{\text{eff}}$ . Hence, in our theory Hawking radiation may automatically shut off as the black hole mass decreases towards its critical value  $M_{\text{crit}}$ , which is in turn determined by when curvature invariants like  $C^2$  reach their limiting values ( $H_0^4$ ).

A consequence of the above is that black hole remnants will remain. Hence, there will be no loss of quantum coherence in the presence of black holes (when calculated in the semiclassical approximation). Neither will there be global charge violation by black holes.

#### 5. Extension to an Anisotropic Universe

Hopefully the reader is at this point persuaded that it is worth while to explore our theory further and see if the wild speculations mentioned in the previous section can indeed be realized.

**Fig. 2:** Phase diagram for the solutions of (3.8), arrows pointing in the direction of increasing time. As can be shown using (3.6), all asymptotic solutions are de Sitter.

As a first step, we have explored<sup>9)</sup> whether our theory can damp out anisotropy at high curvatures, such that asymptotically also an anisotropic Universe will lead to de Sitter space.

Obviously, the action (3.1) with invariant  $I_2$  given by (2.9) is insufficient, since  $I_2$  does not depend on the anisotropy. However, we can easily improve the prospects by changing  $I_2$  to

$$I_2 = 4R_{\mu\nu} R^{\mu\nu} - R^2 + C^2 \quad (5.1)$$

where  $C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$  and  $C_{\mu\nu\rho\sigma}$  is the Weyl tensor. We maintain the form of the action (3.1).

Based on our previous investigations, we should expect to be able to achieve our goal. As  $\varphi_2 \rightarrow \infty$ , the invariant  $I_2$  is again driven to zero. This will imply (in cases when  $C^2 \geq 0$ ) both

$$C^2 = 0 \quad (5.2)$$

and

$$4R_{\mu\nu} R^{\mu\nu} - R^2 = 0. \quad (5.3)$$

The condition (5.2) implies decrease in anisotropy, and then (5.3) tells us that the asymptotic solution (which is homogeneous) will be de Sitter space.

To verify the above claims, we have considered<sup>9)</sup> the simplest anisotropic Universe with metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 e^{\beta(t)} & & \\ & & a^2 e^{\beta(t)} & \\ & & & a^2 e^{-2\beta(t)} \end{pmatrix}. \quad (5.4)$$

The variational equations can be derived using a convenient trick: we replace the time-time component  $g_{00}$  by  $-\alpha(t)^2$ , insert the metric into (3.1) and vary with respect to  $\alpha(t)$ ,  $a(t)$ ,  $\beta(t)$ ,  $\varphi_1(t)$  and  $\varphi_2(t)$ . Still, the resulting equations are rather complicated.

It must be shown that for  $|\varphi_2| \rightarrow \infty$  the anisotropy tends to zero, i.e.  $\dot{\beta} \rightarrow 0$ . This can be done by picking out the terms which dominate in the equations of motion in the limit  $|\varphi_2| \rightarrow \infty$ . As demonstrated in Ref. 9, this is indeed the case.

## Conclusions

We have presented an effective action for gravity based on a higher derivative modification of Einstein's theory of general relativity in which all homogeneous solutions are nonsingular. All corresponding space-time manifolds are geodesically complete and either approach de Sitter space asymptotically or oscillate about Minkowski space. We have speculated that in our theory also singularities inside the black hole horizon might be avoided.

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